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THE GAME PROBLEM ON THE DOLICHOBRACHISTOCHRONE

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The capture and evasion sets, the players' optimal strategies and the game value determined for the game problem on the dolichobrachistochrone, analysed within the framework of a position formalism similar to [1]. Singularities inherent in the game of the minimax-maximin time to contact [1, 2] become apparent; they are determined in the given problem by the specific behavior of the optimal paths close to the target set. Isaacs [4] examined the game problem on the dolichobrachistochrone, being the game analog of the classical variational problem on the brachistochrone [3]. However, as was shown in [5], the solution proposed by Isaacs contains erroneous statements.

1. In the game problem on the dolichobrachistochrone a point m moves in the half-plane of x and y $(y \geqslant 0)$ in accord with the equation

$$x^{\cdot} = \sqrt{y}\cos u + w(v+1)/2, \quad y^{\cdot} = \sqrt{y}\sin u + w(v-1)/2 \tag{1.1}$$

Here w is a positive constant and u and v are control parameters subject to the first and second players, respectively, and to the constraints

$$0 \leqslant u \leqslant 2\pi, \quad -1 \leqslant v \leqslant 1 \tag{1.2}$$

The first player's aim is the most rapid approach of point m the target set

$$M = \{p = \{x, y\} \mid x = 0,\ y \geqslant 0\} \tag{1.3}$$

being positive part of the ordinate axis. The second player tries to prevent point m from hitting onto set M or, at least, to delay it. In the problem statement we assume that point m is in the first quadrant at the initial instant.

In [4] it is stated that for initial points x_0 and y_0 satisfying the conditions $x_0 > 0$ and $0 \leqslant y_0 < w^2$ the second player can prevent approach to the target set M in spite of any efforts of the first player. This statement is justified in [4] in the following manner: in the region $y < w^2$ the second player can move point m arbitrarily far from M by alternately applying the extreme vectors of his vectogram, i.e. by alternately using the extreme values of parameter v (see [4], p. 90). A counterexample is constructed in [5] showing the error in the statement made. It turns out that points exist also in the region $x > 0, 0 \leqslant y < w^2$, from which the first player is able to effect approach to M despite any counteractions of the second player; consequently, the straight line $y = w^2$ cannot be a barrier [4].

2. Let us pose the problem more precisely. Following [1], we identify the first and second players' strategies with functions $u\ (p)$ and $v\ (p)$ of the position $p = \{x, y\}$ satisfying constraints (1.2). The strategy $U \div u\ (p)\ (V \div v\ (p))$ generates a bundle of motions $\Pi\ (p_0,\ U)\ (\Pi\ (p_0,\ V))$ emanating from the position $p_0 = \{x_0,\ y_0\}$ at $t = 0$. According to [1] the motion $p\ [\cdot] \in \Pi\ (p_0,\ U)$ is determined as a function $p\ [t]$ for which we can find, on every finite interval $0 \leqslant t \leqslant \vartheta$, a sequence of Euler polygonal lines $p_{\Delta^{(k)}}\ [t] = \{x_{\Delta^{(k)}}[t],\ y_{\Delta^{(k)}}\ [t]\}$ defined by the conditions

$$\dot{x}_{\Delta^{(k)}}\ [t] = \sqrt{y_{\Delta^{(k)}}\ [t]} \cos u\ (p_{\Delta^{(k)}}\ [\tau_i^{(k)}]) + w\ (v^{(k)}\ [t] + 1) / 2 \qquad (2.1)$$

$$\dot{y}_{\Delta^{(k)}}\ [t] = \sqrt{y_{\Delta^{(k)}}\ [t]} \sin u\ (p_{\Delta^{(k)}}\ [\tau_i^{(k)}]) + w\ (v^{(k)}\ [t] - 1) / 2$$

$$\tau_i^{(k)} \leqslant t < \tau_{i+1}^{(k)}, \quad p_{\Delta^{(k)}}\ [0] = p_0, \quad \tau_0^{(k)} = 0 \quad (i = 1, 2 \ldots)$$

converging uniformly to $p\ [t]$ on the interval $0 \leqslant t \leqslant \vartheta$ and such that $\sup_i (\tau_{i+1}^{(k)} - \tau_i^{(k)}) \to 0$ as $k \to \infty$. We note that $\Delta^{(k)}$ in (2.1) denotes a certain partitioning of the semiaxis $0 \leqslant t < \infty$ into the intervals $\tau_i^{(k)} \leqslant t < \tau_{i+1}^{(k)}$, while $v^{(k)}\ [\cdot]$ denotes a certain measurable function satisfying conditions (1.2). The elements of set $\Pi\ (p_0,\ V)$ are determined analogously. As in [1] we can show that the sets $\Pi\ (p_0,\ U)$ and $\Pi\ (p_0,\ V)$ are not empty and all their elements $p\ [t] = \{x\ [t],\ y\ [t]\}$ are absolutely continuous functions defined for $0 \leqslant t < \infty$ and satisfying the condition $y\ [t] \geqslant 0$. In addition, $\Pi\ (p_0,\ U)$ and $\Pi\ (p_0,\ V)$ have at least one element in common.

For every motion $p\ [t]$ we define $\theta\ (p\ [\cdot])$ as the smallest number $\vartheta \geqslant 0$ for which $p\ [\vartheta] \in M$. If $p\ [\vartheta] \notin M$ for all $\vartheta \in [0, \infty)$, we assume $\theta\ (p\ [\cdot]) = \infty$. Similarly to [1] the pair of strategies U° and V° form a saddle point for position p_0 if

$$\theta\ (p\ [\cdot]) \leqslant T\ (p_0) < \infty \qquad (2.2)$$

for every motion $p\ [\cdot] \in \Pi\ (p_0,\ U^\circ)$ and

$$\theta\ (p\ [\cdot]) \geqslant T\ (p_0) \qquad (2.3)$$

for every motion $p\ [\cdot] \in \Pi\ (p_0,\ V^\circ)$. The quantity $T\ (p_0)$ is called the game's value for the position p_0. If however for a position p_* a strategy U^* exists such that $\theta\ (p\ [\cdot]) < \infty$ for every motion $p\ [\cdot] \in \Pi\ (p_*,\ U^*)$, then in accord with [1] we say that the approach problem is solvable for position p_*. If for a position p_* we can find a strategy V^* and some open neighborhood $H\ (M)$ of set M, such that $p\ [t] \notin H\ (M)$ for $t \in [0, \infty)$ for all $p\ [\cdot] \in \Pi\ (p_*,\ V^*)$, we say that the evasion problem is solvable for position p_*. Below on the basis of the dynamic programing method we determine the capture and evasion sets, i.e. sets of the positions for which the problems of approach and evasion, respectively, are solvable; we also determine the game's value on the capture set.

3. In the case being analyzed the fundamental equation of the dynamic programing method [4] is

$$\min_{u\in[0,2\pi]}\max_{v\in[-1,1]}[(\sqrt{y}\cos u+w(v+1)/2)T_x+ \tag{3.1}$$
$$(\sqrt{y}\sin u+w(v-1)/2)T_y]+1=0$$

Here T_x and T_y, are the partial derivatives in x and y of the function $T(x,y)$. Computing the extremal values in (3.1), we obtain

$$\cos u^\circ=-T_x/\rho,\quad \sin u^\circ=-T_y/\rho,\quad \rho=(T_x^2+T_y^2)^{1/2} \tag{3.2}$$

$$v^\circ=\operatorname{sign}A,\quad A=T_x+T_y \tag{3.2}$$

Using (3.2) and (3.3), Eq. (3.1) takes the form

$$-\sqrt{y}\rho+w(A\operatorname{sign}A+T_x-T_y)/2+1=0 \tag{3.4}$$

In correspondence with the procedure in [4] the boundary condition for Eq. (3.4) is given on an admissible set which we denote $M_F\subset M$ and which is defined by the relation (see [4], p. 92)

$$\min_{u\in[0,2\pi]}\max_{v\in[-1,1]}[\sqrt{y}\cos u+w(v+1)/2]=-\sqrt{y}+w<0 \tag{3.5}$$

From (3.5) we obtain

$$M_F=\{p=\{x,y\}\mid x=0,\ y>w^2\} \tag{3.6}$$

From the meaning of the value of the game we have

$$T(p)=0\quad\text{for}\quad p\in M_F \tag{3.7}$$

The equations for the characteristics for (3.4) are

$$x^\circ=\sqrt{y}T_x/\rho-w(\operatorname{sign}A+1)/2,\quad T_x^\circ=0 \tag{3.8}$$
$$y^\circ=\sqrt{y}T_y/\rho-w(\operatorname{sign}A-1)/2,\quad T_y^\circ=-\rho/(2\sqrt{y})$$
$$(z^\circ=dz/d\tau,\quad \tau=-t)$$

We find T_x and T_y on set M_F from (3.4) and (3.7), which together with the parametric representation of M_F yields the initial conditions for Eqs. (3.8)

$$x(0)=0,\ y(0)=s,\ T_x(0)=(\sqrt{s}-w)^{-1},\ T_y(0)=0,\ s>w^2 \tag{3.9}$$

Equations (3.8) with initial conditions (3.9) are integrated in [1]. The solution obtained is

$$x=\frac{\tau\sqrt{s}}{2}+\frac{s}{2}\sin\frac{\tau}{\sqrt{s}},\quad T_x=(\sqrt{s}-w)^{-1} \tag{3.10}$$

$$y=\frac{s}{2}\left(1+\cos\frac{\tau}{\sqrt{s}}\right),\quad T_y=\frac{\sqrt{s/y-1}}{\sqrt{s}-w},\quad \tau\in\left[0,\frac{\pi\sqrt{s}}{2}\right] \tag{3.11}$$

The quantity A decreases monotonically on each of the characteristics with parameter s, remaining positive up to the instant $\tau_0(s)=\pi\sqrt{s}/2$ at which it vanishes. The points on the characteristics, corresponding to this instant, form a parabola L_0 whose parametric equation is

$$x=\left(\frac{\pi}{4}+\frac{1}{2}\right)s+\frac{\pi w}{2}\sqrt{s},\quad y=\frac{s}{2} \tag{3.12}$$

The parabola L_0 and the characteristics (3.10) are shown in Fig. 1.

When $s \geqslant s_0 = 4\pi^2 w^2/(\pi + 2)^2$ every characteristic satisfies the condition $x(\tau, s) \geqslant 0$ for $\tau \in [0, \tau_0(s)]$. When $s \in (w^2, s_0)$ only a part of each characteristic satisfies the condition $x(\tau, s) \geqslant 0$. By the problem's hypothesis the initial position x_0, y_0 lies in the first quadrant and, therefore, on the basis of the solution of Eq. (3.4) we should use only the parts of the characteristics satisfying the condition $x \geqslant 0$ for determining the game's value.

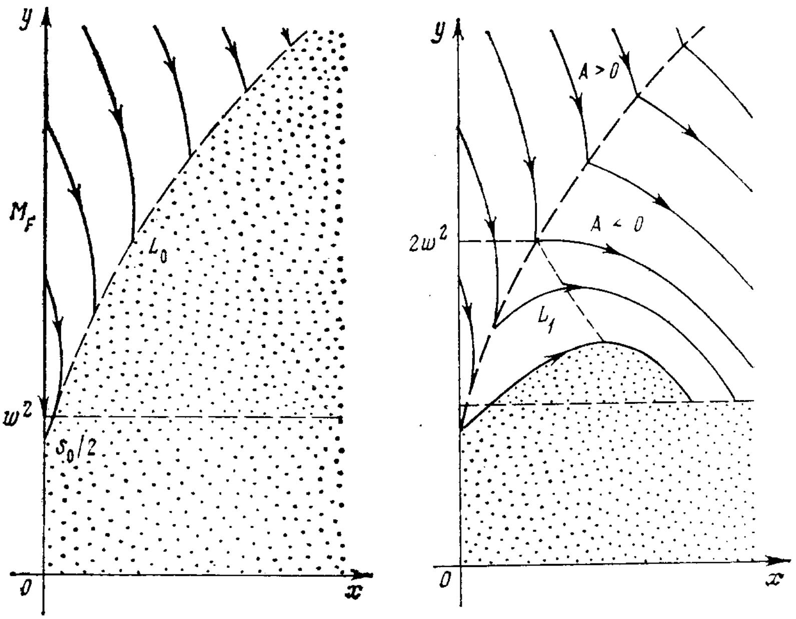

Fig. 1 Fig. 2

In order to continue the integration of Eqs. (3.8) we set $A < 0$ in it. As the initial conditions for the resulting equations we take the values on characteristics (3.10) and (3.11) at the instant $\tau_0(s)$ that the characteristic (3.10) reaches L_0 and we take the parameters of parabola L_0. As a result we have

$$x(\tau_0(s)) = \left(\frac{\pi}{4} + \frac{1}{2}\right)s - \frac{w\pi}{2}\sqrt{s}, \quad y(\tau_0(s)) = \frac{s}{2}$$
$$T_x(\tau_0(s)) = -T_y(\tau_0(s)) = 1/(\sqrt{s} - w), \quad s \in [s_0, \infty) \tag{3.13}$$

We use Eq. (3.4) to separate the variables in (3.8). Soving (3.4) for T_y and taking into account that $T_x = (\sqrt{s} - w)^{-1}$, we obtain two values for T_y

$$T_y = \frac{w \pm Q(y, s)}{w^2 - y}, \quad Q(y, s) = \left[y\left(\frac{w^2 - y}{(\sqrt{s} - w)^2} + 1\right)\right]^{1/2} \tag{3.14}$$

We note that the magnitude of T_y, determined by formula (3.14) in which we have chosen the minus sign, can be extended when $y = w^2$ by continuity by the value $T_y = (1 - w^2 T_x^2)/(2w)$; we assume the satisfaction of this.

From (3.4), with due regard to $A < 0$, we find $\rho = (1 - wT_y)\sqrt{y}$. Using the expression for the quantity ρ and formula (3.14), the Eqs. (3.8) for the characteristics can be transformed to the form

$$x^\circ = \frac{y(y - w^2)}{(y \pm wQ(y, s))(\sqrt{s} - w)}, \quad y^\circ = \pm\frac{Q(y, s)(w^2 - y)}{y \pm wQ(y, s)} \tag{3.15}$$

Here the upper (lower) sign corresponds to the upper (lower) sign in formula (3.14). When the minus sign is chosen the right-hand sides of Eqs. (3.15) will again be considered as extended by continuity when $y = w^2$. From (3.14) we see that function $Q(y, s)$ is defined only when

$$y \in [0,\ w^2 + (\sqrt{s} - w)^2] \tag{3.16}$$

Let $W(s)$ be the set on the x, y-plane, defined by condition (3.16), and let $\Gamma(s)$ be its upper boundary. In order to determine which signs should be chosen in Eqs. (3.15) for continuing the integration, we note that when $y = s/2$ the magnitude of $T_y(y, s)$ must coincide with the initial conditions (3.13). From (3.14) we have

$$T_y\left(\frac{s}{2}, s\right) = \frac{2w(\sqrt{s} - w) \pm \sqrt{s}\,|\sqrt{s} - 2w|}{(2w^2 - s)(\sqrt{s} - w)} \tag{3.17}$$

From (3.17) we see that we should choose the plus sign if $s \in [4w^2, \infty)$ and the minus sign $s \in [s_0, 4w^2)$ in formula (3.14) and in the Eqs. (3.15) corresponding to it if condition

$$T_y(s/2, s) = T_y(\tau_0(s)) = (\sqrt{s} - w)^{-1}$$

is to be satisfied. We note that despite the fact that the magnitude of T_y, defined by formula (3.14), is independent of the choice of sign in (3.14), we should choose the plus sign in Eqs. (3.15) since solutions do not exist for these equations with initial conditions (3.13) when the minus sign is chosen in them.

Using Eqs. (3.15) we can construct a family of characteristics for $s \in (s_0, \infty)$. An exemplary form of characteristics (3.8) with initial conditions (3.13) is shown in Fig. 2. When $s \in [4w^2, \infty)$ the characteristics, being solutions of Eqs. (3.15) in which the plus sign has been chosen, "drop downward" during the time τ. When $s \in [s_0, 4w^2)$ the characteristics, being solutions of Eqs. (3.15) in which now the minus sign has been chosen, "rise upward" during the time τ up to the instant $\tau_1(s)$ at which they reach the upper boundary $\Gamma(s)$ of region $W(s)$, smoothly tangent to it. The instant $\tau_1(s)$ is determined from the condition

$$y(\tau_1(s), s) = w^2 + (\sqrt{s} - w)^2 \tag{3.18}$$

The ends of the characteristics, corresponding to the instant $\tau_1(s)$, form the smooth curve L_1 shown in Fig. 2. The solution of Eqs. (3.15) with the minus sign chosen in them cannot be continued for $\tau \in (\tau_1(s), \infty)$ However, for constructing the characteristics when $\tau \in (\tau_1, (s), \infty)$ we can use Eqs. (3.15) having again chosen the plus sign in them and having selected suitable initial conditions for these equations on curve L_1 so as to preserve the continuity of the functions $x(\tau, s)$, $y(\tau, s)$, $T_x(\tau, s)$ and $T_y(\tau, s)$. It turns out that all the characteristics constructed thus approximate the straight line $y = w^2$ for an unboundedly long time.

The family of characteristics constructed does not completely fill the region $y > w^2$ as is erroneously assumed in [4]. At the same time it is easy to see from Eqs. (1.1) that the approach problem is solvable for any point of the region $y > w^2$. To effect the approach it is sufficient for the first player to use, for instance, the strategy U^* defined by the function $u^*(p)$: $u^*(p) = \pi/2$ when $y \geqslant 4w^2$ and $u^*(p) = 3\pi/4$ when $y < 4w^2$ Meanwhile, the function $T(x, y) = \tau(x, y)$, resulting from solving the characteristics' equations $x = x(\tau, s)$ and $y = y(\tau, s)$ with respect to τ and s is not defined in the whole region $y > w^2$ and, consequently, by the rules of dynamic programing, cannot be used for determining the optimal strategies in the region $y > w^2$. In [4] it is noted that the presence of such a situation in a differential game often indicates the existence of singular solutions, viz., of universal paths. However, a direct check of conditions (7.13.2) of [4] shows that such universal curves do not exist in the present game, and, consequently, the procedures for the further integration of Eq. (3.4), developed in [4] in the presence of universal curves, is inapplicable here. In order to continue the integration of Eq. (3.4) we consider certain auxiliary heuristic arguments by means of which we obtain additional boundary conditions for Eq. (3.4). These conditions enable us to conclude the integration of Eq. (3.4) and to use the function $T(x, y)$ resulting from such an integration to construct the strategies U° and V°.

4. In the auxiliary conditions we shall assume that the number $\Theta(p[\cdot])$ in conditions (2.2) and (2.3) is not determined by the first instant ϑ at which the inclusion $p[\vartheta] \in M$ is accomplished but is given by the relation

$$\Theta(p[\cdot]) = \inf\{\vartheta \geqslant 0 \mid x[\vartheta] < 0\} \tag{4.1}$$

and so determines the first instant that the motion $p[\cdot]$ "penetrates" set M. We note that the condition for motions to attain the target set M, treated as a "penetration", was examined in [4]. We assume further that the function $T(x, y) = \tau(x, y)$, obtained by solving the characteristics' equations $x = x(\tau, s)$ and $y = y(\tau, s)$ with respect to τ and s, is the game's value. From formulas (3.8), (3.10) and (3.11) we see that the quantity $A = T_x + T_y$ is positive in the region located above parabola L_0 and is negative in the region below L_0 and already filled up with characteristics (see Fig. 1). Since $A^\circ = -\rho / (2\sqrt{y}) < 0$, the magnitude of A decreases along each characteristic. But then it is natural to expect that the inequality $A < 0$ is retained in the whole region lying below L_0, in which the game's value exists. Let N_0 denote the point with coordinates $\{0, s_0 / 2\}$, being the point of intersection of L_0 with the ordinate axis, and let M_0 be the part of target set M, determined by the condition

$$M_0 = \{p = \{x, y\} \mid x = 0, y \in [0, s_0 / 2]\} \tag{4.2}$$

Further, suppose that strategies U° and V° satisfying conditions (2.2) and (2.3), wherein the quantity $\Theta(p[\cdot])$ is defined by (4.1), exist for an initial position $p_0 = \{x_0, y_0\} \in M_0$; these strategies are determined by formulas (3.2) and (3.3) in the function $T(x, y)$ in the region wherein the game's value exists. Then for all t for which $y[t] \leqslant s_0 / 2$ the condition $p[t] \in M_0$ is satisfied for every motion

$$p[\cdot] = \{x[\cdot], y[\cdot]\} \in \Pi(p_0, U^\circ) \cap \Pi(p_0, V^\circ)$$

As a matter of fact the motion $p[t]$ cannot fall into the region $x < 0$ because this would contradict the optimality of the second player's actions, who has the possibility of not admitting of intersections with set M_0 by the motions $p[\cdot] \in \Pi(p_0, V^\circ)$ when $y[t] \leqslant s_0 / 2$. On the other hand, from the fact that $A < 0$ in the region below parabola L_0 and from relation (3.3) it follows that $v^\circ(p) = -1$, $(V^\circ \div v(p))$, if point p lies below L_0. From Eqs. (1.1), in its own turn, we see that if the motion $p[\cdot] \in \Pi(p_0, U^\circ) \cap \Pi(p_0, V^\circ)$ falls into the region $x > 0$ and is then forced to move in this region in the negative direction of the ordinate axis, it does not reach the target set M, this contradicts the optimality of the first player's efforts.

By virtue of the condition $p[t] = \{x[t], y[t]\} \in M_0$ for $y[t] \leqslant s_0/2$ every motion $p[\cdot] \in \Pi(p_0, U^\circ) \cap \Pi(p_0, V^\circ)$ satisfies the conditions

$$x^\cdot[t] = 0, \ y^\cdot[t] \in [-\sqrt{2y[t]} - w, \sqrt{2y[t]} + w] \tag{4.3}$$

for all t for which $y[t] \leqslant s_0/2$. It is natural to assume that the motions $p[\cdot] \in \Pi(p_0, U^\circ)$, generated by strategy U° among all solutions of (4.3), are characterized, firstly, by $y^\cdot[t] > 0$ when $y[t] \leqslant s_0/2$ and, secondly, by the motions $p[\cdot]$ reaching the point $N_0 \in M_0$ in minimal time as compared with all solutions $p[\cdot]$ of the inclusion in (4.3). But this signifies that the motions $p[\cdot]$ generated by strategies U° and V° satisfy the equation

$$x^\cdot = 0,\ y^\cdot = \sqrt{2y} - w,\ x[0] = 0,\ y[0] = y_0 \tag{4.4}$$

From (4.4) we see that when $y_0 > w^2/2$ the motion of point $p = \{x, y\}$, described by Eqs. (4.4), takes place in the positive direction of the y-axis, while when $y_0 < w^2/2$, in the negative direction. The point $y_0 = w^2/2$ corresponds to the equilibrium position for Eqs. (4.4). Below we ascertain that through this point on the ordinate axis there passes a barrier, i.e. a curve separating the regions wherein the approach and the evasion problems are solvable. Integrating the second of Eqs. (4.4) from y to $s_0/2$, we find the time of motion from the point $p = \{0, y\}$ to N_0

$$\omega_0(y) = \sqrt{s_0} - \sqrt{y} + w \ln \frac{\sqrt{s_0} - w}{\sqrt{2y} - w} \tag{4.5}$$

$$s_0 = 4\pi^2 w^2 / (\pi + 2)^2, \qquad y \in (w^2/2,\ s_0/2]$$

It is natural to expect that after reaching point N_0 the point p moves up to set M_F along a characteristic with parameter s_0, so that the total time of motion equals

$$w(y) = \omega_0(y) + \pi^2 w / (\pi + 2) \tag{4.6}$$

5. Using function $\omega(y)$ as a boundary condition for Eq. (3.4), we continue the formal integration of this equation by the method of characteristics. The heuristic arguments presented in Sect. 4, will not be used later.

In accord with the procedures in [4] the initial conditions for Eq. (3.8) are determined by the function $\omega(y)$ and by Eq. (3.4) in the following way:

$$T_x(\tau_*(s)) = -T_y(\tau_*(s)) = (\sqrt{s} - w)^{-1} \tag{5.1}$$

$$x(\tau_*(s)) = 0,\ y(\tau_*(s)) = s/2,\ \tau_*(s) = \omega(s/2) \tag{5.2}$$

When integrating (3.8) with initial conditions (5.1) and (5.2) we should take advantage of Eqs. (3.15) by suitably choosing the plus and minus signs in these equations just as we did above. The characteristics obtained by a similar integration are shown in Fig. 3. It turns out that as $s \to w^2$ the characteristics converge to a certain curve B_* whose equation, obtainable from (3.15), is

$$x(y) = -[y(w^2 - y)]^{1/2} - w^2 \arcsin \frac{\sqrt{y}}{w} + w^2\left(\frac{1}{2} - \frac{\pi}{4}\right) \tag{5.3}$$

$$y \in [w^2/2,\ w^2]$$

By B we denote a smooth curve obtained by pasting the curve B_* together with the part of the straight line $y = w^2$ lying in the region $x > w^2/(\pi/4 + 1/2)$, while by D_P and D_E we denote the regions into which the curve B divides the first quadrant, as shown in Fig. 3. Using Eqs. (3.10) and (3.15) we can verify that the family of characteristics constructed uniquely covers region D_P when $s \in (w^2, \infty)$. Since $y(\tau, s)$ and $x(\tau, s)$ are continuous in τ and s when $\tau > 0$ and $s > w^2$, the function $T(x, y) = \tau(x, y)$

obtained by solving the characteristics' equations relative to τ is continuously differentiable in D_P. In addition, the function obtained is a solution of Eq. (3.1) in D_P and satisfies the boundary conditions (3.7) and

$$T(0, y) = \omega(y) \text{ for } y \in (w^2/2, s_0/2)] \tag{5.4}$$

We note as well that $T(p) \to \infty$ as $p \to p_* \in B$, $p \in D_P$. The function $T(x, y)$ constructed can be used to solve the approach-evasion problems and to construct the strategies U° and V°.

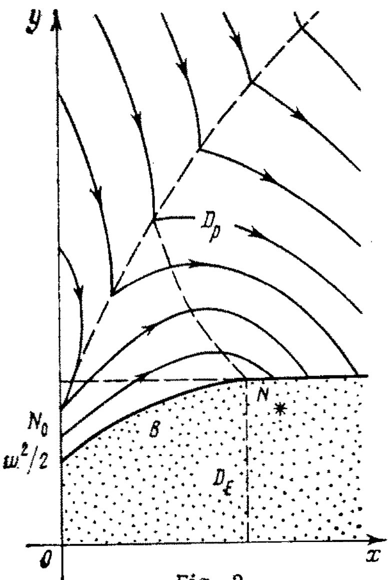

Fig. 3

Let us turn at first to the evasion problem. Using Eq. (8.3.1) in [4] we can show that for Eqs. (1.1) there exists a family of curves called semipermeable in [4] and described by the equation

$$x(y, c) = -[y(w^2 - y)]^{1/2} - w^2 \arcsin \frac{\sqrt{y}}{w} + c \tag{5.5}$$

$$y \in [w^2/2, w^2], \; c \in (-\infty, \infty)$$

The semipermeable curves (5.5) are shown in Fig. 4.

Curves (5.5) possess the following property, called the semipermeability property in [4]. By $G_1, (c_0)$ and $G_2(c_0)$ we denote regions into which a curve (5.5) with parameter c_0 divides the strip (5.6)

$$G = \{p = \{x, y\} \mid y \in [w^2/2, w^2]\}$$

as shown in Fig. 4. Then, if $p_0 \in G_2(c_0)$, the strategy $V^* \div v^*(p)$, $(v^*(p) = -1)$ ensures the preservation of all motions $p[\cdot] = \{x[\cdot], y[\cdot]\} \in \Pi(p_0, V^*)$ in the set $G^{(2)}(c_0)$ for all t for which $y[t] \in [w^2/2, w^2]$. Using the stated property of curves of family (5.5), we can show that the evasion problem is solvable on the set $D_E \cup B^*$, where $B^* = B \setminus B_*$.

By $\eta(p_0)$ we denote the abscissa of the point of intersection of a curve of family (5.5) passing through the point $p_0 \in D_E \cup B^*$ with the straight line $y = w^2/2$, and we introduce the strategy V_{p_0} defined by the function

$$v_{p_0}(p) = v_{p_0}(x, y) = \operatorname{sign}(\eta(p_0) - x) \tag{5.7}$$

Let $E(p_0)$ be a closed set bounded by a curve of family (5.5) passing through point p_0, and by the straight lines $y = w^2$, $x = \eta(p_0)$ and $y = 0$, as shown in Fig. 4. Then, on the basis of the semipermeability property of curves (5.5) we can show that every motion $p[\cdot] \in \Pi(p_0, V_{p_0})$ is preserved on set $E(p_0)$ up to the first instant of hitting onto the closed set

$$E_*(p_0) = \{p = \{x, y\} \mid x \geqslant \eta_*(p_0), \; y \in [0, w^2/2]\}$$

At the same time, turning to (5.7), we can verify that strategy V_{p_0} ensures the satisfaction of the condition $p[t] \in E_*(p_0)$, $t \in [t_*, \infty)$ for all motions $p[\cdot] \in \Pi(p_0, V_{p_0})$ for which $p[t_*] \in E_*(p_0)$. But then for every motion $p[\cdot] \in \Pi(p_0, V_{p_0})$ this same strategy ensures the fulfillment of the inclusion $p[t] \in E(p_0)$, $t \in [0, \infty)$ which jointly with the condition $E(p_0) \cap M^{\eta(p_0)/2} = \varnothing$ implies that strategy V_{p_0} enables all the motions it generates from position p_0 to evade the $\eta(p_0)/2$ neighborhood of set M.

We now consider the approach problem. By N_* we denote the point with coordinates $\{w^2/(\pi/4 + 1/2),\ w^2\}$ at which the curves B_* and B^* are pasted together and by $D_P^{(2)}$ we denote the closed set bounded by curve B and by the characteristic with parameter $s_0 = 4\pi^2 w^2/(\pi+2)^2$, as shown in Fig. 4. Let $D_P^{(1)} = D_P \setminus D_P^{(2)}$. We specify the strategies U° and V° by the functions $u^\circ(p)$ and $v^\circ(p)$ defined by (3.2) and (3.3) when $p \in D_P$ and continued in an arbitrary manner when $p \notin D_P$. For example, we can set $u^\circ(p) = \pi - w^2 \arcsin \sqrt{y}/w$ and $v^\circ(p) = -1$ when $p = \{x, y\} \notin D_P$. The approach problem proves to be solvable on the set $D_P \cup (B_* \setminus N_*)$; in the open set $D_P^{(1)}$ the game's value coincides with the function $T(x, y)$, obtained by the formal integration of Eq. (3.4), while the strategies U° and V° introduced above comply with conditions (2.2) and (2.3) and yield, therefore, a saddle point for the differential game (1.1)–(1.3). Figure 4 depicts the form of motions $p[\cdot] \in \Pi(p_0, U^\circ) \cap \Pi(p_0, V^\circ)$ when $p_0 \in D_P^{(1)}$.

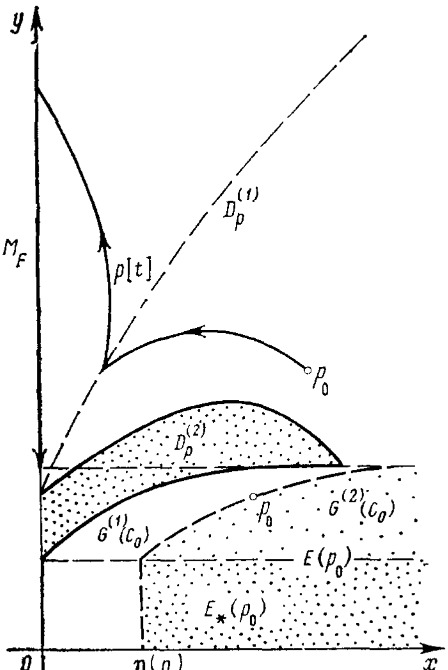

Fig. 4

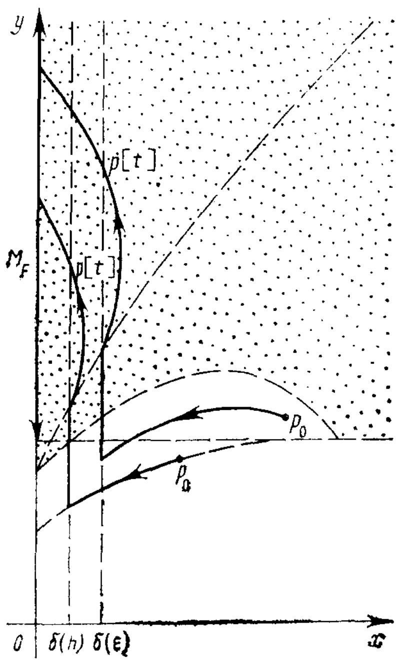

Fig. 5

The structure of the differential game (1.1)–(1.3) turns out to be somewhat different on the set $D_P^{(2)} \cup (B_* \setminus N_*)$. Namely, if $p_0 \in D_P^{(2)} \setminus B$ the strategy U° guarantees the fulfillment of the relation $\theta(p[\cdot]) \leqslant T(p_0)$ for all motions $p[\cdot] \in \Pi(p_0, U^\circ)$. At the same time, a strategy V guaranteeing the satisfaction of condition (2.3) for all motions generated by this strategy does not exist. However, for every $\varepsilon > 0$ we can find a strategy $V_\varepsilon \div v_\varepsilon(p)$ guaranteeing the fulfillment of the relation

$$\theta(p[\cdot]) \geqslant T(p_0) - \varepsilon$$

for all motions $p[\cdot] \in \Pi(p_0, V_\varepsilon)$. For example, we can define the function $v_\varepsilon(p)$ as follows:

$$v_\varepsilon(p) = -1 \text{ for } x \leqslant \delta, \qquad v_\varepsilon(p) = v^\circ(p) \text{ for } x > \delta \tag{5.8}$$

where $\delta = \delta(\varepsilon)$ is a sufficiently small positive number and $v^\circ(p)$ is the function determining strategy V°. Thus, if $p_0 \in D_P^{(2)}$, $p_0 \notin B$ an optimal strategy U° exists for the first player and only an ε-optimal strategy for the second (cf. [1], p. 83). Figure 5 shows the form of the ε-optimal motions $p[\cdot] \in \Pi(p_0, U^\circ) \cap \Pi(p_0, V_\varepsilon)$.

In the case $p_0 \in B_*$, $p_0 \neq N_*$, the strategy U° defined above solves the approach problem for the position p_0; however, the set of instants of first contact of motions $p[\cdot] \in \Pi(p_0, U^\circ)$ with set M turns out to be unbounded in this case, which, as noted in [6], can lead to complications. In the given case this becomes apparent in that although the approach problem is solvable for the position $p_0 = \{x_0, y_0\}$ (all motions $p[\cdot] \in \Pi(p_0, U^\circ) \cap \Pi(p_0, V^\circ)$ reach set M in time $w \ln(w^2 / (2w^2 - y_0))$), nevertheless, for every number h a strategy $V_h \div v_h$ exists guaranteeing the satisfaction of $\theta(p[\cdot]) \geqslant h$ for every motion $p[\cdot] \in \Pi(p_0, V_h)$. The function $v_h(p)$ determining strategy V_h can be defined, for instance, by formulas (5.8) when $p \in D_P$, and $\delta = \delta(h) > 0$ should be chosen sufficiently small, and by the relation

$$v_h(p) = v_h(x, y) = \operatorname{sign}(y + x - w^2 / 2) \tag{5.9}$$

when $p \notin D_P$. By virtue of this the above-described situation of ε-equilibrium, holding for the positions $p_0 \in D_P^{(2)} \setminus B$, vanishes for the positions $p_0 \in B_*$, $p_0 \neq N_*$. Figure 5 shows the form of the motions $p[\cdot] \in \Pi(p_0, U^\circ) \cap \Pi(p_0, V_h)$.

Finally, in the case $p_0 = N_*$, for every second player's strategy and for every open neighborhood $H(M)$ of set M we can find a motion $p[\cdot] \in \Pi(N_*, V)$ falling into $H(M)$ for some $\vartheta \geqslant 0$, so that the evasion problem proves to be unsolvable for the position N_*. At the same time, for every strategy U there exists a motion $p[\cdot] \in \Pi(N_*, U)$ satisfying the condition $p[t] \notin M$ for all $t \in [0, \infty)$, so that the approach problem is also unsolvable for the position N_*. The difficulty mentioned can be successfully overcome if in the evasion problem the condition that every motion evades some neighborhood $H(M)$ of set M is replaced by the condition that every motion evades the set M itself. In this case the evasion problem is solvable for position N_* and the strategy solving the evasion problem indicated can be determined by function (5.14) (sic).

The fundamental assertion made can be proved by estimating the magnitude of the variation of function $T(p)$ on the motions $p[t]$ of system (1.1), similarly as in [1, 7], with due regard to: (1) function $T(p)$ is continuously differentiable in region D_P and jointly with the functions $u^\circ(p)$ and $v^\circ(p)$ satisfies Eq. (3.1) in region D_P; (2) $T(p) > 0$ when $p \notin M$ and $T(p) = 0$ when $p \in M_F$; (3) $p[t] \in D_P$ for every motion $p[\cdot] \in \Pi(p_0, U^\circ)$, $p_0 \in D_P$, in every interval $[0, \vartheta]$ on which $p[t] \notin M$; (4) the evasion problem is solvable in region D_E.

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MAXIMUM PRINCIPLE IN THE PROBLEM OF TIME OPTIMAL RESPONSE WITH NONSMOOTH CONSTRAINTS

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The problem of optimal response [1, 2] with nonsmooth (generally speaking, nonfunctional) constraints imposed on the state variables is considered. This problem is used to illustrate the method of proving the necessary conditions of optimality in the problems of optimal control with phase constraints, based on constructive approximation of the initial problem with constraints by a sequence of problems of optimal control with constraint-free state variables. The variational analysis of the approximating problems is carried out by means of a purely algebraic method involving the formulas for the incremental growth of a functional [3, 4] and the theorems of separability of convex sets is not used.

Using a passage to the limit, the convergence of the approximating problems to the initial problem with constraints is proved, and for general assumptions the necessary conditions of optimality resembling the Pontriagin maximum principle [1] are derived for the generalized solutions of the initial problem. The conditions of transversality are expressed, in the case of nonsmooth (nonfunctional) constraints by a novel concept of a cone conjugate to an arbitrary closed set of a finite-dimensional space. The concept generalizes the usual notions of the normal and the normal cone for the cases of smooth and convex manifolds.

1. Statement of the problem. We consider a general problem of the time optimal response for systems of ordinary differential equations in the class of measurable controls $u(t)$ and absolutely continuous trajectories $x(t)$, $t_0 \leqslant t \leqslant t_1$

$$\dot{x} = f(x, u, t), \quad x = (x^1, \ldots, x^n)' \in \mathbf{R}^n \tag{1.1}$$